Fourier Series:

Are series of cosine and sine terms and arise in the important practical task of representing general **periodic functions.**

Periodic functions:

A function $f(x)$ is called *periodic* if it is defined for all real x and if there is some positive No. T such that

$$f(x+T)=f(x)$$

The No. T is called a period of $f(x)$.

Fourier said If $f(x+T)=f(x)$, T: periodic No. Then

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi}{T}x+b_n\sin\frac{2n\pi}{T}x\right)$$

Where a_0 , a_n & b_n are Fourier coefficients and

$$a_0=\frac{2}{T}\int_A^B f(x)\ dx$$

$$a_n=\frac{2}{T}\int_A^B f(x)\ \cos\frac{2n\pi}{T}x\,dx$$

$$b_n=\frac{2}{T}\int_A^B f(x)\ \sin\frac{2n\pi}{T}x\,dx$$

$$A\prec x\prec B$$

Notes:

$\sin n\pi=0$, $(n=0,\pm1,\pm2,....)$, n integer No.

$$\cos n\pi=\begin{cases}-1 & n=1,3,5,...\\ 1 & n=0,2,4,...\end{cases}$$

$\cos 2n\pi=1$ for all n , $(n=0,\pm1,\pm2,\pm3,....)$

$\cos(-x)=\cos x$ *even*

$\sin(-x)=\sin x$ *odd*

Fourier series:

EX.:

Write Fourier series for $f(x)=x \quad , \quad 0 \prec x \prec 2\pi$

$\Rightarrow \quad T = 2\pi - 0 = 2\pi$

First we find a_0 , a_n & b_n

$$a_0 = \frac{2}{T}\int_A^B f(x)\ dx$$

$$= \frac{2}{2\pi}\int_0^{2\pi} x\ dx = \frac{1}{\pi}\cdot\frac{x^2}{2}\Big|_0^{2\pi} = \frac{1}{2\pi}\left[4\pi^2 - 0\right] = 2\pi$$

$$a_n = \frac{2}{T}\int_A^B f(x)\ \cos\frac{2n\pi}{T}x\,dx$$

$$= \frac{2}{2\pi}\int_0^{2\pi} x\ \cos\frac{2n\pi}{2\pi}x\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\ \cos nx\,dx \qquad , \quad \text{by } u\mathrm{d}v$$

$$= \frac{1}{\pi}\left[x\cdot\frac{1}{n}\ \sin nx\Big|_0^{2\pi} - \int_0^{2\pi}\frac{1}{n}\ \sin nx\,dx\right]$$

$$= \frac{1}{\pi}\cdot\frac{1}{n^2}\cos nx\Big|_0^{2\pi} = \frac{1}{n^2\pi}\left[\cos 2n\pi - \cos 0\right] = \frac{1}{n^2\pi}(1-1) = 0$$

$\therefore\ a_n = 0$

$$b_n = \frac{2}{2\pi}\int_A^B x\ \sin\frac{2n\pi}{2\pi}x\,dx$$

$$= \frac{1}{\pi}\int_A^B x\ \sin nx\,dx \qquad , \quad u = x \quad , \quad dv = \sin nx$$

$$= \frac{1}{\pi}\left[x\cdot(\frac{-1}{n}\cos nx)\Big|_0^{2\pi} - \int_0^{2\pi}\frac{-1}{n}\cos nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[\frac{-1}{n}(2\pi\cdot 1 - 0) + \frac{1}{n^2}\sin nx\Big|_0^{2\pi}\right]$$

$$b_n = \frac{-2}{n}$$

$$b_1 = \frac{-2}{1} \quad , \quad b_2 = \frac{-2}{2} \quad , \quad b_3 = \frac{-2}{3}$$

$$\Rightarrow f(x) = \frac{2}{2\pi} \sum_{n=1}^{\infty} b_n \sin nx \quad , \quad a_n = 0$$

$$= \pi + (b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots)$$

$$= \pi + (\frac{-2}{1} \sin x + \frac{-2}{2} \sin 2x + \frac{-2}{3} \sin 3x + \cdots)$$

$$= \pi - 2(\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \cdots)$$

Fourier even & odd functions

1) If $f(x)$ is even , then

i) $b_n = 0$

ii) $a_0 = \frac{2}{T} \cdot 2 \int_0^B f(x) dx$

iii) $a_n = \frac{2}{T} \cdot 2 \int_0^B f(x) \cos \frac{2n\pi}{T} x \, dx$

2) If $f(x)$ is odd , then

i) $a_0 = a_n = 0$

ii) $b_n = \frac{2}{T} \cdot 2 \int_0^B f(x) \sin \frac{2n\pi}{T} x \, dx$

Def.:

A function $f(x)$ is even if $f(-x) = f(x)$ for all x . For example, $f(x) = x^2$.

A function $f(x)$ is odd if $f(-x) = -f(x)$ for all x . For example, $f(x) = x^3$.

Notes:

- If $f(x)$ symmetric about y-axis $\Rightarrow$ even.

$f(x) = x^2$, $f(x) = \cos x$, $f(x) = |f(x)|$, $f(x) = constant$

- If $f(x)$ symmetric about origin $\Rightarrow$ odd.

$f(x) = \sin x$

EX.:

Write Fourier series for $f(x) = \begin{cases} -1 & , \quad -2 \prec x \prec 0 \\ 1 & , \quad 0 \prec x \prec 2 \end{cases}$

$\Rightarrow T = 2 - (-2) = 4$

i) From sketch $\Rightarrow$ symmetric about origin $\Rightarrow$ odd.

$$\Rightarrow \quad a_0 = a_n = 0$$

$$b_n = \frac{2}{4}\cdot 2\int_0^B 1\cdot \sin\frac{2n\pi}{4}x\,dx$$

$$= \int_0^B 1\cdot \sin\frac{n\pi}{2}x\,dx$$

$$= \frac{-2}{n\pi}\cos\frac{n\pi}{2}x\Big|_0^2 \quad = \frac{-2}{n\pi}(\cos n\pi - 1)$$

$$\therefore\ b_n = \frac{2}{n\pi}(1-\cos n\pi) \quad \ldots.\ (1)$$

To find b_1, put $n = 1$ in eq. (1)

$$\therefore\ b_1 = \frac{2}{\pi}(1-(-1)) = \frac{4}{\pi} \quad , \quad b_2 = \frac{2}{2\pi}(1-1)) = 0$$

$$b_3 = \frac{2}{3\pi}(1+1)) = \frac{4}{3\pi} \quad , \quad b_4 = \frac{2}{5\pi}(1-1)) = 0$$

$$b_5 = \frac{2}{5\pi}(1+1)) = \frac{4}{5\pi}$$

$$\Rightarrow\ f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{2}x$$

$$= \frac{4}{\pi}\sin\frac{\pi}{2}x + 0 + \frac{4}{3\pi}\sin\frac{3\pi}{2}x + 0 + \frac{4}{5\pi}\sin\frac{5\pi}{2}x + \ldots..$$

$$= \frac{4}{\pi}(\sin\frac{\pi}{2}x + \frac{1}{3}\sin\frac{3\pi}{2}x + \frac{1}{5}\sin\frac{5\pi}{2}x + \ldots..)$$

Notes:

$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$
$$\cos(x-y) = \cos x\cos y + \sin x\sin y \quad \text{add}$$

$$\cos(x+y) + \cos(x-y) = 2\cos x\cos y$$

$$\cos x\cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$

We can obtain $\sin x\sin y$ by subtraction.

Fourier series:

$$\sin(x+y)=\sin x\cos y+\cos x\sin y$$
$$\sin(x-y)=\sin x\cos y-\cos x\sin y$$

subtraction

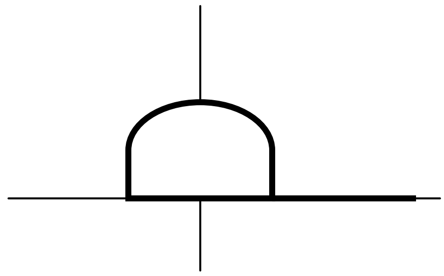
$$\sin(x+y)+\sin(x-y)=2\cos x\sin y$$
$$\cos x\sin y=\frac{1}{2}\sin(x+y)+\sin(x-y)$$

EX.:

Write Fourier series for $f(x)=\begin{cases}\cos x & , \quad \frac{-\pi}{2}\prec x\prec\frac{\pi}{2}\\ 0 & , \quad \frac{\pi}{2}\prec x\prec\frac{3\pi}{2}\end{cases}$

$$\Rightarrow\ T=\frac{3\pi}{2}+\frac{\pi}{2}=2\pi$$

$f(x)$ is even in $\frac{-\pi}{2}\prec x\prec\frac{\pi}{2}$ $\Rightarrow b_n=0$



This is true if and only if the other interval $=0$

$$a_0=\frac{2}{T}\cdot 2\int_0^{\frac{\pi}{2}} f(x)dx=\frac{2}{2\pi}\cdot 2\int_0^{B}\cos x\,dx=\frac{2}{\pi}\ \sin x\Big|_0^{\frac{\pi}{2}}=\frac{2}{\pi}[1-0]=\frac{2}{\pi}$$

$$a_n=\frac{2}{T}\cdot 2\int_0^{\frac{\pi}{2}} f(x)\cos\frac{2n\pi}{2}x\,dx$$

$$=\frac{2}{2\pi}\cdot 2\int_0^{\frac{\pi}{2}}\cos x\cos nx\,dx \qquad ....(1)$$

$$=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{1}{2}[\cos(x+nx)+\cos(x-nx)]\,dx$$

$$=\frac{1}{\pi}\left[\frac{1}{1+n}\sin(x+nx)+\frac{1}{1-n}\sin(x-nx)\right]_0^{\frac{\pi}{2}}$$

$$=\frac{1}{\pi}\left[\frac{1}{1+n}\left\{\sin(\frac{\pi}{2}+n\frac{\pi}{2})-\sin 0\right\}+\frac{1}{1-n}\left\{\sin(\frac{\pi}{2}-n\frac{\pi}{2})-0\right\}\right] \qquad ....(2)$$

To find a_1 , put $n=1$ in eq. (1)

Fourier series:

$$\therefore\ a_1 = \frac{2}{\pi}\cdot\int_0^{\frac{\pi}{2}} \cos x\, dx = \frac{2}{\pi}\cdot\int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2x)\, dx = \frac{1}{\pi}\left[x+\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2}+0-(0+0)\right) = \frac{1}{2}$$

in eq.(2)

$$a_2 = \frac{1}{\pi}\left[\frac{1}{3}\ \{\sin\frac{3\pi}{2}\} - \{\sin\frac{-\pi}{2}\}\right]$$

$$= \frac{1}{\pi}\left(\frac{-1}{3}+1\right) = \frac{2}{3\pi}$$

$$a_3 = \frac{1}{\pi}\left[\frac{1}{4}\ \{0 - \frac{1}{2}\{0\}\}\right] = 0$$

$$a_4 = \frac{1}{\pi}\left[\frac{1}{5}\ \{\sin\frac{5\pi}{2}\} - \frac{1}{3}\{\sin\frac{-3\pi}{2}\}\right]$$

$$= \frac{1}{\pi}\left(\frac{1}{5}-\frac{1}{3}\right) = \frac{-2}{15\pi}$$

$$\Rightarrow\ f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos nx$$

$$= \frac{1}{\pi} + \left(\frac{1}{2}\cos x + \frac{2}{3\pi}\cos 2x - \frac{2}{15\pi}\cos 4x + \ldots\ldots\right)$$

Problems:

Write the Fourier series for the following functions:

1) $f(x) = \begin{cases} a & , \quad 0 \prec x \prec \pi \\ -a & , \quad \pi \prec x \prec 2\pi \end{cases}$

2) $f(x) = |x| \quad , \quad -\pi \prec x \prec \pi$

3) $f(x) = \begin{cases} \pi + x & , \quad -\pi \prec x \prec 0 \\ \pi - x & , \quad 0 \prec x \prec \pi \end{cases}$

4) $f(x) = |\sin x| \quad , \quad -\pi \prec x \prec \pi$

Fourier series:

5) $f(x)=\begin{cases}0 & , \quad -2 \prec x \prec 0 \\ 1 & , \quad 0 \prec x \prec 2\end{cases}$

6) $f(x)=\begin{cases}-1 & , \quad -1 \prec x \prec 0 \\ 2x & , \quad 0 \prec x \prec 1\end{cases}$

7) $f(x)=\begin{cases}k & , \quad \frac{-\pi}{2} \prec x \prec \frac{\pi}{2} \\ 0 & , \quad \frac{\pi}{2} \prec x \prec \frac{3\pi}{2}\end{cases}$

8) $f(x)=\begin{cases}x & , \quad \frac{-\pi}{2} \prec x \prec \frac{\pi}{2} \\ \pi - x & , \quad \frac{\pi}{2} \prec x \prec \frac{3\pi}{2}\end{cases}$

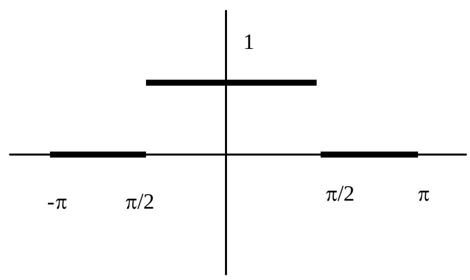
9) $f(x)=x \quad , \quad -\pi \prec x \prec \pi$.

10) $f(x)=x^3 \quad , \quad -\pi \prec x \prec \pi$.

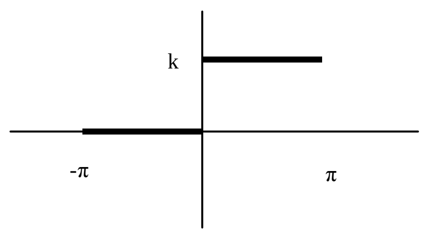
11) $f(x)=\begin{cases}1 & , \quad -\pi \prec x \prec 0 \\ -1 & , \quad 0 \prec x \prec \pi\end{cases}$

12) $f(x)=\begin{cases}x & , \quad \frac{-\pi}{2} \prec x \prec \frac{\pi}{2} \\ 0 & , \quad \frac{\pi}{2} \prec x \prec \frac{3\pi}{2}\end{cases}$

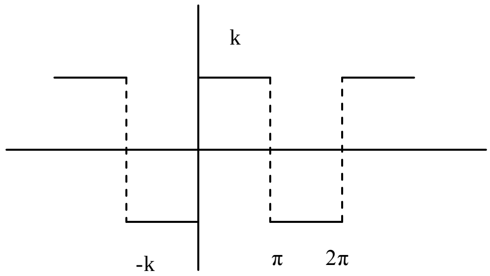
13) Find the Fourier series of the function $f(x)$ which is assumed to have the period 2π .



14) Find the Fourier series of the function $f(x)$



15) Find the Fourier series of the function $f(x)$



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